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Optimal control of European double barrier basket options

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Abstract — We consider European double barrier basket call options on two underlyings with an upper and a lower knock-out barrier featuring a finite number of cash settlements at prespecified values of the underlyings between the strike and the upper barrier. The bilaterally constrained cash settlements are considered as controls that have to be chosen such that the Delta of the option is as close as possible to a predefined constant profit/loss. This leads to a control constrained optimal control problem for the two-dimensional Black-Scholes equation with Dirichlet boundary control and finite time control. Based on the variational formulation of the problem in an appropriate Sobolev space setting, we prove the existence of a unique solution and state the first order necessary optimality conditions. A semi-discretization in space by conforming P1 finite elements with respect to a simplicial triangulation of the computational domain gives rise to a semi-discrete control constrained optimal control problem for a linear system of first order ordinary differential equations. A further discretization in time by the backward Euler scheme results in a fully discrete optimization problem that is solved numerically by the projected gradient method with Armijo line search. Numerical examples for some selected test cases illustrate the benefits of hedging with European double barrier basket options in case of optimally controlled cash settlements.

Keywords: European double barrier basket options, multiple cash settlements, optimal control, Dirichlet and final time control, Black-Scholes equation, finite element discretization

1. Introduction

Options that are different from plain vanilla American or European call or put options are commonly referred to as exotic options (cf., e.g., [16,24,34]). Among the exotic options, those of single or double barrier type are of particular interest. Such options are either activated (knock in options) or expire (knock out options) when the value of the underlying reaches some predetermined upper and/or lower bound (barrier). The valuation of a single barrier option on one underlying has already

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been dealt with in the seminal paper by Merton [22] and subsequently studied in [6,9,27,29]. The first contribution addressing barrier options with more than one underlying is [15]. In particular, the authors are concerned with barrier options on a single stock with the barrier being determined by another asset, whereas valuation formulas for barrier options on a basket have been derived later in [18,33]. Hedging techniques for barrier options have been considered by different approaches including static hedging based on the equivalence to contingent claims with specifically adjusted pay-off functions [7,8,26], the partial differential equation (PDE) formulation [2,11,19,23,26,30], and stochastic optimization [14,20,21].

In this paper, we will follow an optimal control approach for hedging barrier options with multiple cash settlements at the option's expiration [4]. The paper is organized as follows: In section 2, we consider hedging with European double barrier basket call options on two underlyings featuring a finite number of cash settlements at predetermined values of the underlyings between the strike and the upper barrier. The cash settlements are treated as bilaterally constrained control variables that have to be chosen in such a way that a tracking type objective functional in terms of the Delta of the option is minimized. This amounts to the solution of a control constrained optimal control problem for the Black-Scholes equation in some space-time domain $Q := \Omega \times (0, T)$, $T > 0$, where Ω is a trapezoidal domain in \mathbb{R}^2 determined by the lower and upper barriers K_{min} and K_{max} . The cash settlement at the upper barrier occurs as a Dirichlet boundary control, whereas the remaining cash settlements enter as a final time control vector. A particular feature is that the Dirichlet boundary conditions on the boundaries parallel to the coordinate axes are given by the solution of associated 1D Black-Scholes equations. In section 3, we reformulate the problem as an initial control/Dirichlet boundary control problem by means of a simple transformation in time and deal with its weak formulation in a weighted Sobolev space setting. Section 4 is devoted to the derivation of the first order necessary optimality conditions involving adjoint states that satisfy backward in time parabolic PDEs as well as a variational inequality due to the bilateral constraints on the control. In section 5, we are concerned with a semi-discretization in space by conforming P1 finite elements with respect to a simplicial triangulation of the computational domain. The semi-discrete control problem amounts to the minimization of a semi-discrete objective functional subject to systems of first order ordinary differential equations (ODEs) obtained by the finite element approximation in space and the constraints imposed on the controls. It thus represents a control constrained initial control problem for the respective systems of first order ODEs in terms of the associated mass and stiffness matrices as well as input matrices expressing the input from the semi-discretized boundary controls at the upper barrier. The corresponding semi-discrete optimality system nicely reflects the intrinsic couplings between the states, their adjoint counterparts, and the controls. Using a further discretization in time by the implicit Euler method, the resulting fully discrete optimality system is studied in section 6. Its numerical solution is realized by the projected BFGS method. Finally, section 7 contains a documentation of numerical results illustrating the application of the optimal control approach.

2. Hedging with European double barrier basket options

We consider a European double barrier basket call option on a basket consisting of two assets with prices S_i , $1 \leq i \leq 2$. We assume that the option has maturity $T > 0$, strike $K > 0$, and barriers K_{min}, K_{max} satisfying $K_{min} < K < K_{max}$. Hence, the spatial domain $\Omega \subset \mathbb{R}_+^2$ for the price $y(S, t)$, $S = (S_1, S_2) \in \Omega$, $t \in [0, T]$, of the option is given by the trapezoid

$$\Omega := \{S = (S_1, S_2) \in \mathbb{R}_+^2 \mid K_{min} < |S| < K_{max}\}, \quad (2.1)$$

where $|S| := S_1 + S_2$, with boundaries

$$\begin{aligned} \Gamma_1 &:= (K_{min}, K_{max}) \times \{0\} \quad , \quad \Gamma_2 := \{0\} \times (K_{min}, K_{max}), \\ \Gamma_3 &:= \{S \in \mathbb{R}_+^2 \mid |S| = K_{min}\} \quad , \quad \Gamma_4 := \{S \in \mathbb{R}_+^2 \mid |S| = K_{max}\}. \end{aligned} \quad (2.2)$$

We denote by $r = r(t)$, $t \in [0, T]$, the risk-free interest rate and by $\sigma_k = \sigma_k(S, t)$, $1 \leq k \leq 2$, $S \in \Omega$, $t \in [0, T]$, the volatilities of the assets. Moreover, we refer to $\rho = (\rho_{k\ell})_{k,\ell=1}^2$ with $\rho_{kk} = 1$, $1 \leq k \leq 2$, and $\rho_{12} = \rho_{21} = 2\rho/(1 + \rho^2)$, $-1 < \rho < +1$, as the correlation matrix. We set $\xi = (\xi_{k\ell})_{k,\ell=1}^2$ where $\xi_{k\ell} := \rho_{k\ell} \sigma_k \sigma_\ell$, $1 \leq k, \ell \leq 2$. Then, it is well-known (cf., e.g., [1,4,31]) that the price y_Q , $Q := \Omega \times (0, T)$, of the option satisfies the following boundary value problem for the Black-Scholes equation with a final time condition at maturity T :

$$\frac{\partial y_Q}{\partial t} + L_\Omega(t)y_Q = 0 \quad \text{in } Q := \Omega \times (0, T), \quad (2.3a)$$

$$y_Q = y_{\Sigma_j} \quad \text{on } \Sigma_j := \Gamma_j \times (0, T), \quad 1 \leq j \leq 4, \quad (2.3b)$$

$$y_Q(\cdot, T) = y_{Q,T} \quad \text{in } \Omega. \quad (2.3c)$$

Here, $L_\Omega(t)$, $t \in [0, T]$, stands for the second order elliptic operator

$$L_\Omega(t) := \frac{1}{2} \sum_{k,\ell=1}^2 \xi_{k\ell} S_k S_\ell \frac{\partial^2}{\partial S_k \partial S_\ell} + r \sum_{k=1}^2 S_k \frac{\partial}{\partial S_k} - r. \quad (2.4)$$

The payoff $y_{Q,T}$ at maturity T is given by

$$y_{Q,T}(S) := (|S| - K)_+, \quad S \in \Omega. \quad (2.5)$$

Moreover, $y_{\Sigma_3} = 0$ and the constant y_{Σ_4} represents a cash settlement at the upper barrier Σ_4 , whereas y_{Σ_v} , $1 \leq v \leq 2$, have to be computed as the solutions of the one-dimensional Black-Scholes equations

$$\frac{\partial y_{\Sigma_v}}{\partial t} + L_{\Gamma_v}(t)y_{\Sigma_v} = 0 \quad \text{in } \Sigma_v := \Gamma_v \times (0, T), \quad (2.6a)$$

$$y_{\Sigma_v}(S_v, t) = \begin{cases} 0, & S_v = K_{min} \\ y_4, & S_v = K_{max} \end{cases}, \quad t \in (0, T), \quad (2.6b)$$

$$y_{\Sigma_v}(\cdot, T) = y_{Q,T}|_{\Gamma_v} \quad \text{in } \Gamma_v, \quad (2.6c)$$

where $L_{\tau_v}(t)$, $1 \leq v \leq 2$, $t \in [0, T]$, are the second order elliptic operators

$$L_{\tau_v}(t) := \frac{1}{2} \sigma_v^2 S_v^2 \frac{\partial^2}{\partial S_v^2} + r S_v \frac{\partial}{\partial S_v} - r. \quad (2.7)$$

However, for hedging with European double barrier basket options, more complicated payoffs are possible. In fact, given a partitioning

$$K =: K_0 < K_1 < \dots < K_M := K_{\max}, \quad M \in \mathbb{N},$$

of $[K, K_{\max}]$ with $K_i := K + i\delta_{|S|}$, $0 \leq i \leq M$, $\delta_{|S|} := (K_{\max} - K)/M$ and setting formally $u := (u_1, \dots, u_M)^T \in \mathbb{R}_+^M$, as well as $K_{-1} = K_{\min}$, $u_{-1} = u_0 = 0$, we may choose $y_4 = u_M$ and $y_T = g(u)$ where

$$(g(u))(S) = u_{i-1} g_1^{(i)}(S) + u_i g_2^{(i)}(S) \text{ for } |S| \in [K_{i-1}, K_i], \quad i = 0, \dots, M, \quad (2.8)$$

$$g_1^{(i)}(S) := (K_i - S)/\delta_{|S|}, \quad g_2^{(i)}(S) := (S - K_{i-1})/\delta_{|S|}.$$

We may consider u as a control vector that has to be chosen such that the Greek $\Delta := \nabla y$ per asset point is as close to a prespecified profit $d = (d_1, d_2)^T$ as possible. The controls are subject to the constraints

$$u \in U_{ad} := \{v = (v_1, \dots, v_M)^T \in \mathbb{R}_+^M \mid v_i \in U_{ad}^{(i)}, \quad 1 \leq i \leq M\}, \quad (2.9)$$

$$U_{ad}^{(i)} := \{v_i \in \mathbb{R}_+ \mid u_{i,\min} \leq v_i \leq u_{i,\max}\}.$$

We consider the following optimal control problem for the two-dimensional Black-Scholes equation: Find (y_Q, u) such that

$$\inf_{y_Q, u} J(y_Q, u) := \frac{1}{2} \int_0^T \int_{\Omega} |\nabla y_Q - d|^2 dS dt, \quad (2.10)$$

subject to (2.3a)-(2.3c), (2.6a)-(2.6c), and (2.9).

3. Variational formulation of the optimal control problem

We use standard notation from Lebesgue and Sobolev space theory. In particular, given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, with boundary $\Gamma := \partial\Omega$, for $D \subseteq \Omega$ we refer to $L^p(D)$, $1 \leq p \leq \infty$ as the Banach spaces of p -th power integrable functions ($p < \infty$) and essentially bounded functions ($p = \infty$) on D with norm $\|\cdot\|_{L^p(D)}$. We denote by $L^p(D)_+$ the positive cone in $L^p(D)$, i.e., $L^p(D)_+ := \{v \in L^p(D) \mid v \geq 0 \text{ a.e. in } D\}$. In case $p = 2$, the space $L^2(D)$ is a Hilbert space whose inner product and norm will be referred to as $(\cdot, \cdot)_{L^2(D)}$.

For $m \in \mathbb{N}_0$ and weight functions $\omega = (\omega_\alpha)_{|\alpha| \leq m}$ with $\omega_\alpha \in L^\infty(D)_+$, $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$, $|\alpha| := \sum_{i=1}^d \alpha_i$, we denote by $W_\omega^{m,p}(D)$ the weighted Sobolev spaces with norms

$$\|v\|_{W_\omega^{m,p}(D)} := \begin{cases} \left(\sum_{|\alpha| \leq m} \|\omega_\alpha D^\alpha v\|_{L^p(D)}^p \right)^{1/p}, & \text{if } p < \infty \\ \max_{|\alpha| \leq m} \|\omega_\alpha D^\alpha v\|_{L^\infty(D)}, & \text{if } p = \infty \end{cases},$$

and refer to $|\cdot|_{W_\omega^{m,p}(D)}$ as the associated seminorms. In particular, for $|\alpha| = 1$ we use the notation $\nabla \omega v := (S_1 \partial v / \partial S_1, \dots, S_d \partial v / \partial S_d)^T$. For $p < \infty$ and $s \in \mathbb{R}_+$, $s = m + \sigma$, $m \in \mathbb{N}_0$, $0 < \sigma < 1$, we define the weighted Sobolev space $W_\omega^{s,p}(D)$ with norm $\|\cdot\|_{W_\omega^{s,p}(D)}$ in analogy to the standard, non-weighted case and refer to $W_{\omega,0}^{s,p}(D)$ as the closure of $C_0^\infty(D)$ in $W_\omega^{s,p}(D)$. For $s < 0$, we denote by $W_\omega^{-s,p}(D)$ the dual space of $W_{\omega,0}^{-s,q}(D)$, $p^{-1} + q^{-1} = 1$. In case $p = 2$, the spaces $W_\omega^{s,2}(D)$ are Hilbert spaces. We will write $H_\omega^s(D)$ instead of $W_\omega^{s,2}(D)$ and refer to $(\cdot, \cdot)_{H_\omega^s(D)}$ and $\|\cdot\|_{H_\omega^s(D)}$ as the inner products and associated norms. In the standard case $\omega_\alpha \equiv 1$, $|\alpha| \leq m$, we will drop the subindex ω .

For a Banach space X and its dual X^* , we refer to $\langle \cdot, \cdot \rangle_{X^*, X}$ as the dual pairing between X^* and X . For Banach spaces X_i , $1 \leq i \leq n$, $n \in \mathbb{N}$, and a function $v \in \bigcap_{i=1}^n X_i$, we refer to $\|v\|_{\bigcap_{i=1}^n X_i}$ as the norm

$$\|v\|_{\bigcap_{i=1}^n X_i} := \max_{1 \leq i \leq n} \|v\|_{X_i}. \quad (3.1)$$

Moreover, for $T > 0$ and a Banach space X , we denote by $L^p((0, T), X)$, $1 \leq p \leq \infty$, and $C([0, T], X)$ the Banach spaces of functions $v : [0, T] \rightarrow X$ with norms

$$\|v\|_{L^p((0, T), X)} := \begin{cases} \left(\int_0^T \|v(t)\|_X^p dt \right)^{1/p}, & 1 \leq p < \infty \\ \text{ess sup}_{t \in [0, T]} \|v(t)\|_X, & p = \infty \end{cases}, \quad \|v\|_{C([0, T], X)} := \max_{t \in [0, T]} \|v(t)\|_X.$$

The spaces $W^{s,p}((0, T), X)$ and $H^s((0, T), X)$, $s \in \mathbb{R}_+$, are defined likewise.

In particular, for a subspace $V \subset H_\omega^1(\Omega)$ with dual V^* we will consider the space

$$H^1((0, T), V^*) \cap L^2((0, T), V), \quad (3.2)$$

and note that the following continuous embedding holds true

$$H^1((0, T), V^*) \cap L^2((0, T), V) \subset C([0, T], L^2(\Omega)). \quad (3.3)$$

For $y \in H^1((0, T), V^*) \cap L^2((0, T), V)$, we further denote by $\gamma_{\Sigma'}(y)$, $\Sigma' \subset \Sigma := \Gamma \times (0, T)$, the trace of y on Σ' .

In the sequel, $\Omega \subset \mathbb{R}_+^2$ will stand for the trapezoidal domain given by (2.1) and

$\Gamma_i \subset \partial\Omega, 1 \leq i \leq 4$, for its boundaries as specified by (2.2). The weight functions ω_α will be chosen according to

$$w_\alpha = S_{\alpha_1} S_{\alpha_2}, \quad \alpha = (\alpha_1, \alpha_2), \quad |\alpha| \leq 2, \quad (3.4)$$

with the convention that $S_{\alpha_i} = 1$ for $\alpha_i = 0, 1 \leq i \leq 2$.

We reformulate the boundary and final time problems for the backward parabolic equations as initial-boundary value problems by means of the transformation $t \mapsto T - t$. Keeping for notational simplicity the same notation for y_Q and y_{Σ_v} and the operators $L_\Omega, L_{\Sigma_v}, 1 \leq v \leq 2$, the initial-boundary value problems read as follows:

$$\frac{\partial y_Q}{\partial t} - L_\Omega(t)y_Q = 0 \quad \text{in } Q := \Omega \times (0, T), \quad (3.5a)$$

$$y_Q = \begin{cases} y_v, & \text{on } \Sigma_v := \Gamma_v \times (0, T), \quad 1 \leq v \leq 2, \\ 0, & \text{on } \Sigma_3 := \Gamma_3 \times (0, T) \\ u_M, & \text{on } \Sigma_4 := \Gamma_4 \times (0, T) \end{cases}, \quad (3.5b)$$

$$y_Q(\cdot, 0) = g(u) \quad \text{in } \Omega, \quad (3.5c)$$

$$\frac{\partial y_{\Sigma_v}}{\partial t} - L_{\Gamma_v}(t)y_{\Sigma_v} = 0 \quad \text{in } \Sigma_v, \quad (3.6a)$$

$$y_{\Sigma_v}(S_v, t) = \begin{cases} 0, & S_v = K_{\min} \\ u_M, & S_v = K_{\max} \end{cases}, \quad t \in (0, T), \quad (3.6b)$$

$$y_{\Sigma_v}(\cdot, 0) = g(u)|_{\Gamma_v} \quad \text{in } \Gamma_v. \quad (3.6c)$$

As far as the volatilities $\sigma_k, 1 \leq k \leq 2$, and the interest r are concerned, we will impose the following assumptions:

(A1) The volatilities satisfy $\sigma_k \in C([0, T], C^2(\bar{\Omega})), 1 \leq k \leq 2$, and there exist constants $\sigma_k^{(\min)} > 0, C_{\sigma_k} > 0$, such that

$$\sigma_k(S, t) \geq \sigma_k^{(\min)}, \quad (S, t) \in \bar{Q}, \quad 1 \leq k \leq 2, \quad (3.7a)$$

$$|S \cdot \nabla \sigma_k(S, t)| \leq C_{\sigma_k}, \quad (S, t) \in \bar{Q}, \quad 1 \leq k \leq 2. \quad (3.7b)$$

(A2) The interest rate satisfies $r \in C([0, T])$ such that $r(t) > 0, t \in [0, T]$.

For the correlation matrix $\xi = (\xi_{k,\ell})_{k,\ell=1}^2$, it is an immediate consequence of assumption **(A1)** that $\xi_{k,\ell} \in C([0, T], C^2(\bar{\Omega})), 1 \leq k, \ell \leq 2$, and that there exists a constant $\xi_{\min} > 0$ such that for all $\eta \in \mathbb{R}^2$ there holds

$$\sum_{k,\ell=1}^2 \xi_{k,\ell}(S, t) \eta_k \eta_\ell \geq \xi_{\min} |\eta|^2, \quad (S, t) \in \bar{Q}. \quad (3.8)$$

We now study the weak formulations of the initial-boundary value problems (3.5a)-(3.5c) and (3.6a)-(3.6c):

A function $y_Q \in H^1((0, T), V^*) \cap L^2((0, T), V)$, where $V := \{v \in H_\omega^1(\Omega) \mid v|_{\Sigma_{mu}} = y_{\Sigma_v}, 1 \leq v \leq 2, v|_{\Sigma_3} = 0, v_{\Sigma_4} = u_M\}$, is called a weak solution of (3.5a)-(3.5c), if for all $v \in L^2((0, T), H_{\omega,0}^1(\Omega))$ there holds

$$\int_0^T \left\langle \frac{\partial y_Q}{\partial t}, v \right\rangle_{H_\omega^{-1}(\Omega), H_{\omega,0}^1(\Omega)} dt + \int_0^T a(t; y_Q, v) dt = 0, \quad (3.9a)$$

$$y_Q(\cdot, 0) = g(u) \quad (3.9b)$$

Here, the bilinear form $a(t; \cdot, \cdot), t \in (0, T)$, is given by

$$a(t; y, v) := \int_\Omega \left(\frac{1}{2} \sum_{k,\ell=1}^2 \xi_{k\ell} S_k \frac{\partial y}{\partial S_k} S_\ell \frac{\partial v}{\partial S_\ell} - \sum_{k=1}^2 r S_k \frac{\partial y}{\partial S_k} v - \left(\frac{1}{2} \sum_{k,\ell=1}^2 (S_k S_\ell \frac{\partial \xi_{k\ell}}{\partial S_\ell} + \xi_{k\ell} S_k) - r \right) y v \right) dS.$$

Likewise, a function $y_{\Sigma_v} \in H^1((0, T), V_v^*) \cap L^2((0, T), V_v)$, where $V_v := \{v \in H_\omega^1(\Sigma_v) \mid v(K_{min}) = 0, v(K_{max}) = u_M\}$, is said to be a weak solution of (3.6a)-(3.6c), if for all $v_v \in L^2((0, T), H_{\omega,0}^1(\Gamma_v))$ there holds

$$\int_0^T \left\langle \frac{\partial y_{\Sigma_v}}{\partial t}, v_v \right\rangle dt + \int_0^T a_v(t; y_{\Sigma_v}, v_v) dt = 0, \quad (3.10a)$$

$$y_{\Sigma_v}(\cdot, 0) = g(u)|_{\Gamma_v}. \quad (3.10b)$$

Here, the bilinear form $a_v(t; \cdot, \cdot), t \in (0, T)$, is given by

$$a_v(t; y, v) := \int_{\Gamma_v} \left(\frac{1}{2} \sigma_v^2 S_v \frac{\partial y}{\partial S_v} S_v \frac{\partial v}{\partial S_v} - r S_v \frac{\partial y}{\partial S_v} v - (\sigma_v^2 S_v + \sigma_v S_v^2 \frac{\partial \sigma_v}{\partial S_v} - r) y v \right) dS_v.$$

Theorem 3.1. *For arbitrarily chosen but fixed $u \in \mathbb{R}^M$, the state equations (3.9a), (3.9b) and (3.10a), (3.10b) admit unique solutions $y_Q \in C([0, T], V) \cap L^2((0, T), V \cap H_\omega^2(\Omega))$ and $y_{\Sigma_v} \in C([0, T], V_v) \cap L^2((0, T), V_v \cap H_\omega^2(\Gamma_v)), 1 \leq v \leq 2$. Moreover, for all $t \in (0, T)$ there holds*

$$\exp(-2\lambda t) \|y_Q(t)\|_{L^2(\Omega)}^2 + 2\xi_{min}^2 \int_0^t \exp(-2\lambda \tau) |y_Q(\tau)|_V^2 d\tau \leq \|g(u)\|_{L^2(\Omega)}^2, \quad (3.11)$$

$$\exp(-2\lambda_v t) \|y_{\Sigma_v}(t)\|_{L^2(\Gamma_v)}^2 + \frac{1}{2} (\sigma_v^{(min)})^2 \int_0^t \exp(-2\lambda_v \tau) |y_{\Sigma_v}(\tau)|_{V_v}^2 d\tau \leq \|g(u)\|_{L^2(\Gamma_v)}^2, \quad 1 \leq v \leq 2.$$

Proof. It follows from assumptions $(\mathbf{A}_1), (\mathbf{A}_2)$ and the Poincaré inequalities (cf., e.g., [1])

$$\begin{aligned} \|v\|_{L^2(\Omega)}^2 &\leq 2 \int_{\Omega} |S \cdot \nabla_{\omega} v(S)|^2 dS_1 dS_2, \quad v \in V, \\ \|v_{\nu}\|_{L^2(\Gamma_{\nu})}^2 &\leq 2 \int_{\Gamma_{\nu}} |S_{\nu} \frac{\partial v_{\nu}}{\partial S_{\nu}}(S_{\nu})|^2 dS_{\nu}, \quad v_{\nu} \in V_{\nu}, \quad 1 \leq \nu \leq 2, \end{aligned}$$

that the bilinear forms $a(t; \cdot, \cdot)$ and $a_{\nu}(t; \cdot, \cdot)$ satisfy Gårding-type inequalities uniformly in t , i.e., there exist constants $\lambda \geq 0$ and $\lambda_{\nu} \geq 0, 1 \leq \nu \leq 2$, such that for all $t \in (0, T)$ there holds

$$\begin{aligned} a(t; v, v) &\geq \frac{1}{4} \xi_{\min}^2 \|v\|_V^2 - \lambda \|v\|_{L^2(\Omega)}^2, \quad v \in V, \\ a_{\nu}(t; v, v) &\geq \frac{1}{4} (\sigma_{\nu}^{(\min)})^2 \|v\|_{V_{\nu}}^2 - \lambda_{\nu} \|v\|_{L^2(\Gamma_{\nu})}^2, \quad v \in V_{\nu}, \quad 1 \leq \nu \leq 2. \end{aligned}$$

Consequently, the initial-boundary value problems (3.9a),(3.9b) and (3.10a),(3.10b) have unique solutions $y_Q \in H^1((0, T), V^*) \cap L^2((0, T), V)$ and $y_{\Sigma_{\nu}} \in H^1((0, T), V_{\nu}^*) \cap L^2((0, T); V_{\nu}), 1 \leq \nu \leq 2$, satisfying (3.11) (cf., e.g., Thm. 2.11 and section 2.6 in [1]). Moreover, standard regularity results for parabolic partial differential equations [1, 28] reveal $y_Q \in C([0, T], V) \cap L^2((0, T), V \cap H_{\omega}^2(\Omega))$ and $y_{\Sigma_{\nu}} \in C([0, T], V_{\nu}) \cap L^2((0, T), V_{\nu} \cap H_{\omega}^2(\Gamma_{\nu})), 1 \leq \nu \leq 2$. \square

Based on the weak formulation of the state equations, the optimal control problem from section 2 reads as follows:

Find (y, u) , where $y = (y_Q, y_{\Sigma_1}, y_{\Sigma_2}), y_Q \in H^1((0, T), V^*) \cap L^2((0, T), V), y_{\Sigma_{\nu}} \in H^1((0, T), V_{\nu}^*) \cap L^2((0, T); V_{\nu}), 1 \leq \nu \leq 2$, and $u \in U_{ad}$ such that

$$\inf_{y, u} J(y, u) := \frac{1}{2} \int_0^T \int_{\Omega} |\nabla y_Q - d|^2 dS dt, \quad (3.12a)$$

$$\text{subject to (3.9a),(3.9b) and (3.10a),(3.10b).} \quad (3.12b)$$

Theorem 3.2. *The optimal control problem (3.12a),(3.12b) admits a unique solution (y, u) .*

Proof. We denote by $S : U_{ad} \rightarrow H^1((0, T), V^*) \cap L^2((0, T), V)$ and $S_{\nu} : U_{ad} \rightarrow H^1((0, T), V_{\nu}^*) \cap L^2((0, T); V_{\nu}), 1 \leq \nu \leq 2$, the control-to-state maps which assign to

an admissible control $u \in U_{ad}$ the unique solutions y_Q and y_{Σ_v} , $1 \leq v \leq 2$, of the state equations (3.9a),(3.9b) and (3.10a),(3.10b). Replacing $y = (y_Q, y_{\Sigma_1}, y_{\Sigma_2})$ in (3.12a) with $(S(u), S_1(u), S_2(u))$, the reduced formulation of (3.12a), (3.12b) is given by:

$$\inf_{u \in U_{ad}} J^{red}(u) := \frac{1}{2} \int_0^T \int_{\Omega} |\nabla S(u) - d|^2 dSdt, \quad (3.13a)$$

such that the triple $(S(u), S_1(u), S_2(u))$ satisfies (3.9a),(3.9b) and (3.10a),(3.10b). (3.13b)

Let $(u_n)_{n \in \mathbb{N}}, u_n \in U_{ad}, n \in \mathbb{N}$, be a minimizing sequence. Due to the facts that $(u_n)_{n \in \mathbb{N}}$ is bounded and $U_{ad} \subset \mathbb{R}^M$ is closed, there exist a subsequence $\mathbb{N}' \subset \mathbb{N}$ and $u^* \in U_{ad}$ such that $u_n \rightarrow u^*$ ($\mathbb{N}' \ni n \rightarrow \infty$). From the continuity of the control-to-state maps we deduce

$$S(u_n) \rightarrow S(u^*) \quad , \quad S_v(u_n) \rightarrow S_v(u^*) \quad (\mathbb{N}' \ni n \rightarrow \infty).$$

Moreover, $(S(u^*), S_1(u^*), S_2(u^*))$ satisfies (3.9a),(3.9b) and (3.10a),(3.10b). Taking additionally the continuity of g into account, we find

$$J^{red}(u_n) \rightarrow J^{red}(u^*) \quad (\mathbb{N}' \ni n \rightarrow \infty),$$

which allows to conclude. □

4. Necessary optimality conditions

The first order necessary optimality conditions can be stated in terms of $(y, u), y = (y_Q, y_{\Sigma_1}, y_{\Sigma_2})$, and an adjoint state $p \in W_0(0, T)$ that is the solution of a final time problem on Q with homogeneous Dirichlet boundary conditions.

Theorem 4.1. *If $(y, u) \in W(0, T) \times U_{ad}$ is the optimal solution of (3.12a),(3.12b), there exists $p_Q \in W_0(0, T)$ such that there holds:*

(i) p_Q is the weak solution of the parabolic final time problem

$$-\frac{\partial p_Q}{\partial t} - A^* p_Q = -\nabla \cdot (\nabla y_Q - d) \quad \text{in } Q, \quad (4.1a)$$

$$p_Q = 0 \quad \text{on } \Sigma, \quad (4.1b)$$

$$p_Q(\cdot, T) = 0 \quad \text{in } \Omega, \quad (4.1c)$$

(ii) *The variational inequality*

$$\left(\int_0^T \left(\gamma_{\Sigma_4}(n_{\Sigma_4} \cdot R_{\Sigma_4}(p_Q)) \right) dt - g_u^*(u) p_Q(0) \right) \cdot (v - u) \geq 0, \quad v \in U_{ad}. \quad (4.2)$$

is satisfied, where $R_{\Sigma_4}(p)$ is given by

$$R_{\Sigma_4}(p_Q) = (R_{\Sigma_4}^{(1)}(p_Q), R_{\Sigma_4}^{(2)}(p_Q))^T, \quad R_{\Sigma_4}^{(k)}(p_Q) := S_k \left(\frac{1}{2} \sum_{\ell=1}^2 \xi_{k\ell} S_\ell \frac{\partial p_Q}{\partial S_\ell} - r p_Q \right), \quad 1 \leq k \leq 2, \quad (4.3)$$

and $g_u^*(u) \in \mathcal{L}(L^2(\Omega), \mathbb{R}^M)$ stands for the adjoint of the Fréchet derivative of g at $u \in U_{ad}$.

Proof. We introduce multipliers

$$p = (p_Q, (p_{\Sigma_v})_{v=1}^2), \quad q = ((q_{\Sigma_v})_{v=1}^4, (q_{K_{max},v})_{v=1}^2, (q_{K_{min},v})_{v=1}^2, q_{0,\Omega}, (q_{0,\Gamma_v})_{v=1}^2),$$

such that

$$\begin{aligned} p_Q &\in W_0(0, T), \quad p_{\Sigma_v} \in W_{v,0}(0, T), \quad 1 \leq v \leq 2, \\ q_{\Sigma_v} &\in L^2((0, T), H_\omega^{-1/2}(\Gamma_v)), \quad 1 \leq v \leq 4, \quad q_{K_2,v}, q_{K_1,v} \in \mathbb{R}, \quad 1 \leq v \leq 2, \\ q_{0,\Omega} &\in L^2(\Omega), \quad q_{0,\Gamma_v} \in L^2(\Gamma_v), \quad 1 \leq v \leq 2. \end{aligned}$$

We consider the Lagrangian

$$\begin{aligned} L(y, u, p, q) &:= J(y_Q, u) + \int_0^T \left\langle \frac{\partial y_Q}{\partial t} - A(t)y_Q, p_Q \right\rangle dt + \sum_{v=1}^2 \int_0^T \left\langle \frac{\partial y_{\Sigma_v}}{\partial t} - A_v(t)y_{\Sigma_v}, p_{\Sigma_v} \right\rangle dt \\ &+ \sum_{v=1}^4 \int_0^T \langle q_{\Sigma_v}, y_{\Sigma_v} - \gamma_{\Sigma_v}(y_Q) \rangle dt + \sum_{v=1}^2 \int_0^T \left(q_{K_{max},v} (u_M - \gamma_{K_{max},v}(y_{\Sigma_v})) - q_{K_{min},v} \gamma_{K_{min},v}(y_{\Sigma_v}) \right) \\ &+ (y_Q(0) - g(u), q_{0,\Omega})_{L^2(\Omega)} + \sum_{v=1}^2 (y_{\Sigma_v}(0) - g(u), q_{0,\Gamma_v})_{L^2(\Gamma_v)}, \end{aligned}$$

where $\gamma_{\Sigma_v}(y_Q)$, $1 \leq v \leq 2$, is the trace of y_Q on Σ_v and

$$\gamma_{K,v}(y_{\Sigma_v}) = y_{\Sigma_v}(K, \cdot), K \in \{K_{min}, K_{max}\}, 1 \leq v \leq 2.$$

Denoting by $A_v^*(t)$ the adjoint of $A_v(t)$, $1 \leq v \leq 2$, and introducing

$$R_K(p_v) := \frac{1}{2} \sigma_v^2 S_K^2 \frac{\partial p_{\Sigma_v}}{\partial S_v}(K) - r S_K p_{\Sigma_v}(K), \quad 1 \leq v \leq 2, \quad K \in \{K_{min}, K_{max}\},$$

integration by parts yields in time and an application of Green's formula results in

$$\begin{aligned}
L(y, u, p, q) = & \quad (4.4) \\
J(y, u) + \int_0^T \left\langle -\frac{\partial p_Q}{\partial t} - A^*(t)p_Q, y_Q \right\rangle dt + \sum_{v=1}^2 \int_0^T \left\langle -\frac{\partial p_{\Sigma_v}}{\partial t} - A_v^*(t)p_{\Sigma_v}, y_{\Sigma_v} \right\rangle dt + \\
& \sum_{v=1}^4 \int_0^T \left(\langle \gamma_{\Sigma_v}(n_{\Sigma_v} \cdot R_{\Sigma_v}(p_Q)) - q_{\Sigma_v}, \gamma_{\Sigma_v}(y_Q) \rangle + \langle q_{\Sigma_v}, y_{\Sigma_v} \rangle \right) dt + \\
& \sum_{v=1}^2 \int_0^T \left((R_{K_{max}}(p_{\Sigma_v}) - q_{K_{max},v}) \gamma_{K_{max},v}(y_{\Sigma_v}) + q_{K_{max},v} u_M \right) dt - \\
& \sum_{v=1}^2 \int_0^T \left((R_{K_{min}}(p_{\Sigma_v}) - q_{K_{min},v}) \gamma_{K_{min},v}(y_{\Sigma_v}) \right) dt + \\
& (y_Q(0), q_{0,\Omega} - p_Q(0))_{L^2(\Omega)} - (g(u), q_{0,\Omega})_{L^2(\Omega)} + (y_Q(T), p_Q(T))_{L^2(\Omega)} + \\
& \sum_{v=1}^2 \left((y_{\Sigma_v}(0), q_{0,\Sigma_v} - p_{\Sigma_v}(0))_{L^2(\Sigma_v)} - (g(u), q_{0,\Sigma_v})_{L^2(\Sigma_v)} + (y_{\Sigma_v}(T), p_{\Sigma_v}(T))_{L^2(\Sigma_v)} \right).
\end{aligned}$$

Here, $R_{\Sigma_v}(p_Q)$, $1 \leq v \leq 3$, is defined as in (4.3) with Σ_4 replaced by Σ_v , $1 \leq v \leq 3$. In view of $J_y(y_Q, u) = -\nabla \cdot (\nabla y_Q - d)$, the optimality conditions

$$L_y(y, u, p, q) = 0 \quad \text{and} \quad L_q(y, u, p, q) = 0$$

reveal that

$$q_{\Sigma_v} = \gamma_{\Sigma_v}(n_{\Sigma_v} \cdot R_{\Sigma_v}(p)), \quad 1 \leq v \leq 4, \quad (4.5a)$$

$$q_{K_{min},v} = R_{K_{min}}(p_v) \quad , \quad q_{K_{max},v} = R_{K_{max}}(p_v), \quad 1 \leq v \leq 2, \quad (4.5b)$$

$$q_{0,\Omega} = \gamma_{0,\Omega}(p) \quad , \quad q_{0,\Sigma_v} = p_v(0), \quad 1 \leq v \leq 2, \quad (4.5c)$$

and

$$\gamma_{\Sigma_v}(y_Q) = y_{\Sigma_v}, \quad 1 \leq v \leq 4, \quad (4.6a)$$

$$\gamma_{K_{min},v}(y_{\Sigma_v}) = 0 \quad , \quad \gamma_{K_{max},v}(y_{\Sigma_v}) = u_M, \quad 1 \leq v \leq 2, \quad (4.6b)$$

$$y_Q(\cdot, 0) = g(u), \quad y_{\Sigma_v}(\cdot, 0) = g(u)|_{\Sigma_v}, \quad 1 \leq v \leq 2. \quad (4.6c)$$

Further, p_{Σ_v} , $1 \leq v \leq 2$, is the weak solution of

$$\begin{aligned}
-\frac{\partial p_{\Sigma_v}}{\partial t} - A_v^*(t)p_{\Sigma_v} &= n_{\Sigma_v} \cdot R_{\Sigma_v}(p_Q) \quad \text{in } \Sigma_v, \\
R_K(p_{\Sigma_v}) &= 0, \quad K \in \{K_{min}, K_{max}\}, \\
p_{\Sigma_v}(\cdot, T) &= 0.
\end{aligned}$$

Since $n_{\Sigma_v} \cdot R_{\Sigma_v}(p_Q) = 0, 1 \leq v \leq 2$, it follows that $p_{\Sigma_v} = 0$ and hence $p_{\Sigma_v} = \gamma_{\Sigma_v}(p_Q), 1 \leq v \leq 2$.

Taking (4.5a)-(4.5c) and (4.6a)-(4.6c) into account, the optimality condition

$$L_p(y, u, p, q) = 0$$

shows that y_Q and $y_{\Sigma_v}, 1 \leq v \leq 2$, are the weak solutions of (3.5a)-(3.5c) and (3.6a)-(3.6c). Finally, observing (2.8) and $y_{\Sigma_4} = u_M$ as well as the regularity results of Theorem 3.1, the optimality condition

$$(L_u(y, u, p, q)) \cdot (v - u) \geq 0 \quad , \quad v \in U_{ad},$$

gives rise to (4.2). \square

5. Semi-discretization of the optimal control problem

The parabolic problems (3.9a),(3.9b) and (3.10a),(3.10b) will be discretized in space by conforming P1 finite elements. To this end, we consider a shape-regular simplicial triangulation $\mathcal{T}_h(\Omega)$ of Ω which aligns with $\Gamma_j, 1 \leq j \leq 4$, so that this triangulation generates triangulations $\mathcal{T}_h(\Gamma_j)$ of $\Gamma_j, 1 \leq j \leq 4$, as well. We refer to $\mathcal{N}_h(D)$ and $\mathcal{E}_h(D)$, $D \subseteq \bar{\Omega}$, as the sets of vertices in $D \subseteq \bar{\Omega}$. We denote by h_T and $|T|$ the diameter and area of an element $T \in \mathcal{T}_h^{(m)}(\Omega)$. For $D \subset \bar{\Omega}$, we refer to $P_k(D), k \in \mathbb{N}_0$, as the linear spaces of polynomials of degree $\leq k$ on D .

We define V_h as the finite element space of continuous P1 finite elements associated with the triangulation $\mathcal{T}_h(\Omega)$, i.e.,

$$V_h := \{v_h \in C(\bar{\Omega}) \mid v_h|_K \in P_1(K), K \in \mathcal{T}_h(\Omega)\}, \quad (5.1)$$

and we set $V_{h,0} := V_h \cap C_0(\bar{\Omega})$. Likewise, we define $V_{h,v}, 1 \leq v \leq 2$, as the finite element spaces of continuous P1 finite elements associated with the triangulations $\mathcal{T}_h(\Gamma_v)$ attaining the values 0 at $S_v = K_{min}$ and u_M at $S_v = K_{max}$, i.e.,

$$V_{h,v} := \{v_h \in C(\bar{\Gamma}_v) \mid v_h|_K \in P_1(K), K \in \mathcal{T}_h(\Gamma_v), v_h(K_{min}) = 0, v_h(K_{max}) = u_M\}, \quad (5.2)$$

and we define $V_{h,v,0}$ in the same way, but replacing u_M with 0.

The semi-discrete approximation of (3.9a),(3.9b) amounts to the computation of $y_{h,Q} \in C^1([0, T], V_h)$ with $y_{h,Q}(\cdot, t)|_{\Gamma_v} = y_{h,\Gamma_v}(\cdot, t), 1 \leq v \leq 2$, and $y_{h,Q}(\cdot, t)|_{\Gamma_3} = 0, y_{h,Q}(\cdot, t)|_{\Gamma_4} = u_M$, such that

$$\left(\frac{dy_{h,Q}}{dt}, v_h\right)_{L^2(\Omega)} + a(t; y_{h,Q}, v_h) = 0, \quad v_h \in V_{h,0}, \quad (5.3a)$$

$$(y_{h,Q}(\cdot, 0), v_h)_{L^2(\Omega)} = (g(u), v_h)_{L^2(\Omega)}, \quad v_h \in V_h. \quad (5.3b)$$

On the other hand, for the semi-discrete approximation of (3.10a),(3.10b) we have to compute $y_{h,\Gamma_v} \in C^1([0, T], V_{h,v})$, $1 \leq v \leq 2$, such that

$$\left(\frac{dy_{h,\Gamma_v}}{dt}, v_h\right)_{L^2(\Gamma_v)} + a(t; y_{h,\Gamma_v}, v_h) = 0, \quad v_h \in V_{h,v,0}, \quad (5.4a)$$

$$(y_{h,\Gamma_v}(\cdot, 0), v_h)_{L^2(\Gamma_v)} = (g(u), v_h)_{L^2(\Gamma_v)}, \quad v_h \in V_{h,v}. \quad (5.4b)$$

The semi-discrete optimal control problems reads: Find (y_h, u) , where $y_h = (y_{h,Q}, y_{h,\Gamma_1}, y_{h,\Gamma_2})$ such that

$$\inf_{y_h, u} J_h(y_h, u) := \frac{1}{2} \int_0^T \sum_{K \in \mathcal{T}_h(\Omega)} \|\nabla(y_{h,Q}(\cdot, t) - d)\|_{L^2(K)}^2 dt, \quad (5.5a)$$

$$\text{subject to } (5.4a), (5.4b), (5.3a), (5.3b) \text{ and } (2.9). \quad (5.5b)$$

The semi-discrete problem (5.5a),(5.5b) will be written in a more compact form as a control constrained optimal control problem for an initial-value problem associated with a system of first order linear ordinary differential equations. To this end, we set

$$N_\Omega := \text{card}(\mathcal{N}_h(\Omega)), \quad N_{\Gamma_v} := \text{card}(\mathcal{N}_h(\Gamma_v)), \quad 1 \leq v \leq 4, \quad N := N_\Omega + N_{\Gamma_1} + N_{\Gamma_2},$$

and denote by $\varphi_{h,\Omega}^{(i)}, 1 \leq i \leq N_\Omega, \varphi_{h,\Gamma_v}^{(i)}, 1 \leq i \leq N_{\Gamma_v}$, the nodal basis functions associated with the nodal points in $\mathcal{N}_h(\Omega)$ and $\mathcal{N}_h(\Gamma_v), v \in \{1, 2, 4\}$. We introduce

$$M = \begin{pmatrix} M_\Omega & 0 & 0 \\ 0 & M_{\Gamma_1} & 0 \\ 0 & 0 & M_{\Gamma_2} \end{pmatrix}, \quad A(t) = \begin{pmatrix} A_\Omega(t) & A_{\Omega\Gamma_1}(t) & A_{\Omega\Gamma_2}(t) \\ 0 & A_{\Gamma_1}(t) & 0 \\ 0 & 0 & A_{\Gamma_2}(t) \end{pmatrix}, \quad t \in (0, T],$$

as the block mass matrix $M \in \mathbb{R}^{N \times N}$ and the block stiffness matrix $A \in \mathbb{R}^{N \times N}$. Here, $M_{\Omega\Omega} \in \mathbb{R}^{N_\Omega \times N_\Omega}$ and $M_{\Gamma_v\Gamma_v}, 1 \leq v \leq 2$, stand for the matrices

$$(M_\Omega)_{ij} := (\varphi_{h,\Omega}^{(j)}, \varphi_{h,\Omega}^{(i)})_{L^2(\Omega)}, \quad 1 \leq i, j \leq N_\Omega,$$

$$(M_{\Gamma_v})_{ij} := (\varphi_{h,\Gamma_v}^{(j)}, \varphi_{h,\Gamma_v}^{(i)})_{L^2(\Gamma_v)}, \quad 1 \leq i, j \leq N_{\Gamma_v}, \quad 1 \leq v \leq 2,$$

whereas the matrices $A_\Omega(t) \in \mathbb{R}^{N_\Omega \times N_\Omega}, A_{\Omega\Gamma_v}(t) \in \mathbb{R}^{N_\Omega \times N_{\Gamma_v}}, A_{\Gamma_v}(t) \in \mathbb{R}^{N_{\Gamma_v} \times N_{\Gamma_v}}$ are given by

$$(A_\Omega)_{ij} := a(t; \varphi_{h,\Omega}^{(j)}, \varphi_{h,\Omega}^{(i)}), \quad 1 \leq i, j \leq N_\Omega,$$

$$(A_{\Omega\Gamma_v})_{ij} := a(t; \varphi_{h,\Gamma_v}^{(j)}, \varphi_{h,\Omega}^{(i)}), \quad 1 \leq i \leq N_\Omega, \quad 1 \leq j \leq N_{\Gamma_v}, \quad 1 \leq v \leq 2,$$

$$(A_{\Gamma_v})_{ij} := a(t; \varphi_{h,\Gamma_v}^{(j)}, \varphi_{h,\Gamma_v}^{(i)}), \quad 1 \leq i, j \leq N_{\Gamma_v}, \quad 1 \leq v \leq 2.$$

We further introduce the input matrices

$$\begin{aligned}
B(t) &\in \mathbb{R}^{N \times M}, \quad B(t) = (0 \ B_M(t)), \quad 0 \in \mathbb{R}^{N \times (M-1)}, \\
B_M(t) &= (B_{M,\Omega}(t), B_{M,\Gamma_1}(t), B_{M,\Gamma_2}(t))^T, \quad t \in (0, T], \\
B_{M,\Omega}(t) &\in \mathbb{R}^{N_\Omega \times 1}, \quad (B_{M,\Omega}(t))_i := - \sum_{j=1}^{N_{\Gamma_4}} a(t; \varphi_{h,\Gamma_4}^{(j)}, \varphi_{h,\Omega}^{(i)}), \quad 1 \leq i \leq N_\Omega, \\
B_{M,\Gamma_\nu}(t) &\in \mathbb{R}^{N_{\Gamma_\nu} \times 1}, \quad (B_{M,\Gamma_\nu}(t))_i := -a(t; \varphi_{h,\Gamma_4}^{N_{\Gamma_4}^{(\nu)}}, \varphi_{h,\Gamma_\nu}^{(i)}), \quad 1 \leq i \leq N_{\Gamma_\nu}, \quad 1 \leq \nu \leq 2,
\end{aligned}$$

where $N_{\Gamma_4}^{(\nu)} := (2 - \nu) + (\nu - 1)N_{\Gamma_4}$, $1 \leq \nu \leq 2$, and

$$\begin{aligned}
G &\in \mathbb{R}^{N \times M}, \quad G = (G_\Omega, G_{\Gamma_1}, G_{\Gamma_2})^T, \quad G_\Omega \in \mathbb{R}^{N_\Omega \times M}, \quad G_{\Gamma_\nu} \in \mathbb{R}^{N_{\Gamma_\nu} \times M}, \\
(G_\Omega)_{ij} &:= \int_{\Omega_j} g_2^{(j)}(S) \varphi_{h,\Omega}^{(i)}(S) dS + \int_{\Omega_{j+1}} g_1^{(j+1)}(S) \varphi_{h,\Omega}^{(i)}(S) dS, \\
(G_{\Gamma_\nu})_{ij} &:= \int_{K_{j-1}}^{K_j} g_2^{(j)}(S_\nu) \varphi_{h,\Gamma_\nu}^{(i)}(S_\nu) dS_\nu + \int_{K_j}^{K_{j+1}} g_1^{(j+1)}(S_\nu) \varphi_{h,\Gamma_\nu}^{(i)}(S_\nu) dS_\nu, \quad 1 \leq \nu \leq 2.
\end{aligned}$$

$$C_\Omega \in \mathbb{R}^{N_\Omega \times N_\Omega}, \quad (C_\Omega)_{ij} := \sum_{T \in \mathcal{T}_h(\Omega)} \int_T \nabla \varphi_{h,\Omega}^{(j)} \cdot \nabla \varphi_{h,\Omega}^{(i)} dS, \quad 1 \leq i, j \leq N_\Omega,$$

$$D_\Omega^{(k)} \in \mathbb{R}^{N_\Omega \times N_\Omega}, \quad (D_\Omega^{(k)})_{ij} := \sum_{T \in \mathcal{T}_h(\Omega)} \int_T \frac{\partial \varphi_{h,\Omega}^{(j)}}{\partial S_k} \varphi_{h,\Omega}^{(i)} dS, \quad 1 \leq i, j \leq N_\Omega, \quad 1 \leq k \leq 2.$$

The semi-discrete optimal control problem reads as follows: Find $y \in C^1([0, T], \mathbb{R}^N)$, $y = (y_Q, y_{\Sigma_1}, y_{\Sigma_2})$, $u \in \mathcal{U}_{ad}$, such that

$$\inf_{y,u} J(y,u) := \frac{1}{2} \int_0^T \left(y_Q^T C_\Omega y_Q - 2 \sum_{k=1}^2 d_k^T D_\Omega^k y_Q + \sum_{k=1}^2 d_k^T M_\Omega d_k \right) dt, \quad (5.6a)$$

subject to

$$M \frac{dy}{dt} + A(t)y = Bu, \quad t \in [0, T], \quad (5.6b)$$

$$My(0) = Gu. \quad (5.6c)$$

Theorem 5.1. *The semi-discrete optimization problem (5.6a),(5.6b) admits a unique solution. If $y \in C^1([0, T], \mathbb{R}^N)$, $u \in \mathcal{U}_{ad}$ is the optimal solution, there exists $p_Q \in C^1([0, T], \mathbb{R}^N)$ such that*

$$M_\Omega \frac{dp_Q}{dt} - A(t)^T p_Q = -C_\Omega y_Q + \frac{1}{2} \sum_{k=1}^2 (D_\Omega^{(k)})^T d_k, \quad t \in [0, T], \quad (5.7a)$$

$$M_\Omega p_Q(T) = 0, \quad (5.7b)$$

holds true and the variational inequality

$$\left(-G^T p_Q(0) - \int_0^T B_\Omega(t)^T p_Q dt \right) \cdot (v - u) \geq 0, \quad v \in \mathcal{U}_{ad}, \quad (5.7c)$$

is satisfied, where $B_\Omega(t) = (0B_{M,\Omega}(t))$, $0 \in \mathbb{R}^{(M-1) \times N_\Omega}$.

Proof. We introduce multipliers

$$p_Q \in C^1([0, T], \mathbb{R}^N), \quad p_{\Sigma_v} \in C^1([0, T], \mathbb{R}^{N_{\Gamma_v}}, 1 \leq v \leq 2, \quad q_0 \in \mathbb{R}^N,$$

and consider the Lagrangian

$$L(y, u, p, q_0) := J(y, u) + \int_0^T p \cdot (M \frac{dy}{dt} + Ay - Bu) dt + q_0 \cdot (My(0) - Gu).$$

Integration by parts reveals

$$\begin{aligned} L(y, u, p, q_0) &= J(y, u) + \int_0^T y \cdot \left(-M \frac{dp}{dt} + A^T p \right) dt + - \int_0^T p \cdot Bu dt \\ &\quad + y(T) \cdot Mp(T) - y(0) \cdot Mp(0) + y(0) \cdot Mq_0 - q_0 \cdot Gu. \end{aligned}$$

The optimality condition $L_y(y, u, p, q_0) = 0$ gives $q_0 = p_Q(0)$ and shows that p_Q satisfies (5.7a),(5.7b). Moreover, the optimality condition $L_{q_0}(y, u, p, q_0) = 0$ yields $My(0) = Gu$ and together with $L_p(y, u, p, q_0) = 0$ implies that y satisfies (5.6a)-(5.6c). Finally, $L_u(y, u, p, q_0) = 0$ results in (5.7c). \square

6. Fully discrete optimal control problem

For the discretization in time of the semi-discrete optimal control problem (5.6a)-(5.6c) we consider a partition

$$0 =: t_0 < t_1 < \dots < t_R := T/R, \quad R \in \mathbb{N},$$

of the time interval $[0, T]$ with step lengths $\tau_r := t_r - t_{r-1}, 1 \leq r \leq R$. We approximate the ODE (5.6b) by the backward Euler scheme, split the integral in the objective functional (5.6a) into the sum over the subintervals (t_{r-1}, t_r) and use the quadrature formula $\int_{t_{r-1}}^{t_r} v dt \approx \tau_r v(t_r)$. Denoting by $y^r = (y_Q^r, y_{\Sigma_1}^r, y_{\Sigma_2}^r)^T$ approximations of $y = (y_Q, y_{\Sigma_1}, y_{\Sigma_2})^T$ at $t_r, 0 \leq r \leq R$, and setting $y := (y^0, \dots, y^R)^T, y_Q := (y_Q^0, \dots, y_Q^R)^T, y_{\Sigma_v} := (y_{\Sigma_v}^0, \dots, y_{\Sigma_v}^R)^T, 1 \leq v \leq 2$, the fully discrete optimal control problem reads: Find $(y, u) \in \mathbb{R}^{(R+1)N} \times U_{ad}$ such that

$$\inf_{y, u} J(y, u) := \frac{1}{2} \sum_{r=1}^R \tau_r \left((y_Q^r)^T C_{\Omega} y_Q^r - 2 \sum_{k=1}^2 d_k^T D_{\Omega}^{(k)} y_Q^r + \sum_{k=1}^2 d_k^T M_{\Omega} d_k \right), \quad (6.1a)$$

subject to

$$M y^r + \tau_r A(t_r) y^r = \tau_r B u + M y^{r-1}, \quad 1 \leq r \leq R, \quad (6.1b)$$

$$M y^0 = G u. \quad (6.1c)$$

Theorem 6.1. *The fully discrete optimization problem (6.1a)-(6.1c) admits a unique solution. If $y \in \mathbb{R}^{(R+1)N}, u \in U_{ad}$ is the optimal solution, there exists $p_Q = (p_Q^0, \dots, p_Q^R)^T \in \mathbb{R}^{(R+1)N_{\Omega}}$ such that*

$$M_{\Omega} p_Q^{r-1} + \tau_r A_{\Omega}(t_{r-1})^T p_Q^{r-1} = M_{\Omega} p_Q^r + \tau_r (C_{\Omega} y_Q^r + \sum_{k=1}^2 (D_{\Omega}^{(k)})^T d_k), \quad (6.2a)$$

$$M_{\Omega} p_Q^R = 0, \quad (6.2b)$$

and

$$\left(-G_{\Omega}^T p_Q^0 - \sum_{r=0}^{R-1} \tau_{r+1} B(t_r)^T p_Q^r \right) \cdot (v - u) \geq 0, \quad v \in U_{ad}. \quad (6.2c)$$

Proof. The proof is the discrete analogue of Theorem 5.1. \square

7. Fully discrete optimal control problem

We denote by $S : U_{ad} \rightarrow \mathbb{R}^{(R+1)N}$ the control-to-state map which assigns to an admissible control $u \in U_{ad}$ the solution $y \in \mathbb{R}^{(R+1)N}$ of the discrete state equation (6.1a), (6.1b). Then, the so-called control-reduced form of the fully discrete optimal control problem (6.1a)-(6.1c) reads

$$\inf_{u \in U_{ad}} J^{red}(u), \quad J^{red}(u) := J(S(u), u). \quad (7.1)$$

It follows from Theorem 6.1 that the gradient of the control-reduced objective functional is given by

$$\nabla J^{red}(u) = -G_{\Omega}^T p_Q^0 - \sum_{r=0}^{R-1} \tau_{r+1} B_{\Omega}(t_r)^T p^r. \quad (7.2)$$

Denoting by $P_{U_{ad}} : \mathbb{R}^M \rightarrow U_{ad}$ the pointwise projection onto the admissible control set U_{ad} , i.e.,

$$P_{U_{ad}}(w) = \begin{cases} u_{i,min} & , \text{if } w_i \leq u_{i,min} \\ w_i & , \text{if } u_{i,min} \leq w_i \leq u_{i,max} \\ u_{i,max} & , \text{if } w_i \geq u_{i,max} \end{cases}, \quad 1 \leq i \leq M, \quad (7.3)$$

and given an initial control $u^{(0)} \in U_{ad}$, we solve (6.1a)-(6.1c) by the projected gradient method with Armijo line search (cf., e.g., [17,25])

$$u^{(\ell+1)} = P_{U_{ad}}(u^{(\ell)} - \alpha_{\ell} \nabla J^{red}(u^{(\ell)})), \quad \ell \geq 0. \quad (7.4)$$

Here, α_{ℓ} is the step length satisfying the Wolfe conditions

$$J^{red}(u^{(\ell)} - \alpha_{\ell} \nabla J^{red}(u^{(\ell)})) \leq J^{red}(u^{(\ell)}) - c_1 \alpha_{\ell} \|\nabla J^{red}(u^{(\ell)})\|^2, \quad (7.5a)$$

$$\nabla J^{red}(u^{(\ell)})^T \nabla J^{red}(u^{(\ell)} - \alpha_{\ell} \nabla J^{red}(u^{(\ell)})) \leq c_2 \|\nabla J^{red}(u^{(\ell)})\|^2, \quad (7.5b)$$

with $0 < c_1 \ll c_2 < 1$. We note that (7.5a) is called the Armijo rule [3], whereas (7.5b) is referred to as the curvature condition. This leads to the following algorithm:

Algorithm: Projected Gradient Method with Armijo Line Search

Step 1: Choose a tolerance $TOL > 0$ and an initial control $u^{(0)} \in U_{ad}$. Set $\ell := 0$.

Step 2: Compute $y^{(\ell+1)} = S(u^{(\ell)}) \in \mathbb{R}^{(R+1)N}$ and $p_Q^{(\ell+1)} \in \mathbb{R}^{(R+1)N_{\Omega}}$ as the solutions of (6.1b), (6.1c) and (6.2a), (??) (with u in (6.1c) replaced with $u^{(\ell)}$). Update the control by computing $u^{(\ell+1)}$ according to (7.4) with steplength α_{ℓ} chosen by means of (7.5a), (7.5b).

Step 3: If the termination criterion $\|\nabla J^{red}(u^{(\ell)})\| < TOL$ is satisfied, stop the algorithm and accept $(y^{(\ell+1)}, u^{(\ell+1)}, p_Q^{(\ell+1)})$ as approximate solution of the fully discrete optimal control problem. Otherwise, set $\ell := \ell + 1$ and go to Step 2.

8. Numerical results

We provide a documentation of computational results for the optimal control of European double barrier basket call options based on the numerical solution of the optimal control problem as outlined in section 5 and section 6. We have considered the case of $M = 5$ controls, i.e., $u = (u_1, \dots, u_5)^T$ and various values of $d = (d_1, d_2)^T$ in the tracking-type objective functional of the optimal control problem. In the

first example, we have used constant volatilities σ_1, σ_2 and interest rate r , whereas Example 2 deals with the case of variable data. Finally, in order to illustrate the benefits of the optimal control approach, Example 3 deals with a non-optimized European double barrier basket call option featuring a single cash-settlement at the upper barrier.

Example 1. In our first example we have chosen $d = (0.2, 0.2)^T$ in the objective functional. The complete data are given in Table 1 below.

Parameter	Notation	Value
d	Desired Delta	$(0.2, 0.2)^T$
M	Number of controls	5
K_{min}	Lower Barrier	50
K_{max}	Upper Barrier	150
K	Strike	100
T	Maturity	1
r	Interest Rate	0.05
σ_1	Volatility of asset 1	0.35
σ_2	Volatility of asset 2	0.20
ρ	Correlation between assets	-0.5
$u_{i,min}$	Lower bound on the controls	0.0
$u_{i,max}$	Upper bound on the controls	50.0

Table 1. Example 1: Data of the optimal control problem

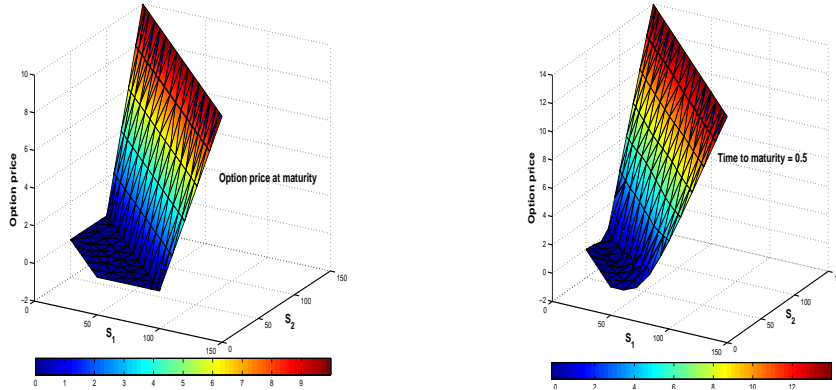


Figure 1. Example 1: Option price at maturity (left) and at time to maturity $t = 0.5$ (right).

For discretization in space, we have chosen a simplicial triangulation of Ω with $h = 5.0$ for both the state and the adjoint state and for discretization in time we

have used a uniform time step of $\Delta t = 0.01$. The projected gradient method with Armijo line search has been initialized with an initial control $u_0 = (0, 50, 0, 50, 0)^T$ and has been stopped when the projected gradient became smaller than $TOL := 1.0E - 06$. The iteration terminated after 48 iterations with the optimal control $u^* = (2, 4, 6, 8, 10)^T$.

ℓ	$J_{red}(u^{(\ell)})$	$\ \nabla J_{red}(u^{(\ell)})\ $	ℓ	$J_{red}(u^{(\ell)})$	$\ \nabla J_{red}(u^{(\ell)})\ $
1	2.5175E+03	9.1334E+01	10	3.9161E+01	4.3928E+00
20	2.4413E+00	6.4258E-02	30	2.4400E+00	4.3687E-03
40	2.4400E+00	1.2411E-04	48	2.4400E+00	4.0830E-07

Table 2. Example 1 (Convergence history (maturity $t = T$)): Number ℓ of projected gradient iteration, value $J_{red}(u^{(\ell)})$ of the objective functional, and norm $\|\nabla J_{red}(u^{(\ell)})\|$ of the gradient.

Table 2 contains a documentation of the convergence history of the projected gradient algorithm with Armijo line search. Here, ℓ stands for the iteration number, $J_{red}(u^{(\ell)})$ is the corresponding value of the objective functional, and $\|\nabla J_{red}(u^{(\ell)})\|$ refers to the norm of the gradient. As a termination criterion for the iteration, we have used $\|\nabla J_{red}(u^{(\ell)})\| < TOL := 1.0E - 06$.

Example 2. The second example deals with the case of space-varying volatilities σ_1, σ_2 , and time-varying interest rate r . The complete data are given in Table 3 below.

Parameter	Notation	Value
d	Desired Delta	$(0.5, 0.5)^T$
M	Number of controls	5
K_{min}	Lower Barrier	50
K_{max}	Upper Barrier	150
K	Strike	100
T	Maturity	1
r	Interest Rate	$r(t) = 0.02 \cdot t + 0.08 \cdot (1 - t)$
σ_1	Volatility of asset 1	$\sigma_1(S, t) = 0.75 \cdot ((S_1 + S_2 - 100)/50)^2$
σ_2	Volatility of asset 2	$\sigma_2(S, t) = 0.75/2 \cdot ((S_1 + S_2 - 100)/50)^2$
ρ	Correlation between assets	-0.5
$u_{i,min}$	Lower bound on the controls	0.0
$u_{i,max}$	Upper bound on the controls	50.0

Table 3. Example 2: Data of the optimal control problem

We have used the same discretizations and the same initial control as in the first example. As tolerance for the termination criterion we have used $TOL = 10^{-6}$. The computed optimal control is $u^* = (17.50, 18.98, 25.07, 29.40, 35.77)^T$.

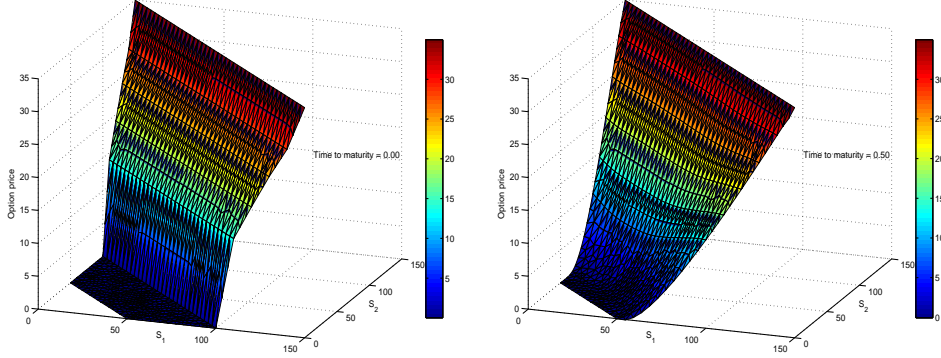


Figure 2. Example 2: Option price at maturity (left) and at time to maturity $t = 0.5$ (right).

ℓ	$J_{red}(u^{(\ell)})$	$\ \nabla J_{red}(u^{(\ell)})\ $	ℓ	$J_{red}(u^{(\ell)})$	$\ \nabla J_{red}(u^{(\ell)})\ $
1	1.5717E+03	4.1822E+01	10	1.0445E+02	2.2505E+00
20	4.7608E+01	2.2148E-02	30	4.7412E+01	3.2269E-04
40	4.7410E+01	5.9259E-06	46	4.7410E+01	8.3484E-07

Table 4. Example 2 (Convergence history (maturity $t = T$)): Number ℓ of the projected gradient iteration, value $J_{red}(u^{(\ell)})$ of the objective functional, and norm $\|\nabla J_{red}(u^{(\ell)})\|$ of the gradient.

The value of the value of the objective functional at optimality is significantly higher as in the first example due to the higher values of d_1 and d_2 . On the other hand, the projected gradient method with Armijo line search performed similarly. Indeed, the termination criterion was reached after 46 iterations.

Example 3. In order to illustrate the benefits of optimized versus non-optimized European double barrier basket call options, we present the numerical results for a non-optimized call option with a single cash settlement of 10 at the upper barrier, but otherwise the same data as in the previous example.

In case the option is still in the money at maturity, its price corresponds to that of a plain vanilla European call option, as can be seen in Figure 3 (left). Otherwise, there are significant differences as displayed in Figure 3 (right). In fact, in comparison with European double barrier basket call options featuring optimized cash settlements and aiming at a constant Delta, the Delta is varying considerably and can even take negative values.

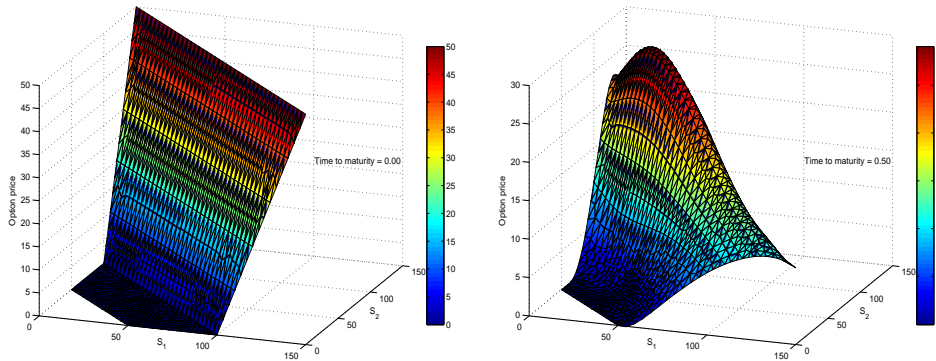


Figure 3. Example 3: Option price at maturity (left) and time to maturity $t = 0.5$ (right).

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